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Rotational Symmetry and Dirac's Monopole

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The field of a Dirac monopole is constructed in the framework of symplectic mechanics by imposing rotational and time translation invariance on the motion of a test particle. Quantization is achieved by the geometric method of Kostant and Souriau, which allows for an elegant solution of the quantum symmetry problem. Space-reflection symmetry is studied in addition.

1. INTRODUCTION

It was pointed out by Hraskó and Balog (1978) that the field of a Dirac monopole can be constructed by imposing rotational (and time translation) symmetry on the motion of a test particle. Their paper is closely related to a previous one (Frenkel and Hraskó, 1977), where it is found that the rotation and parity operators for a particle in a monopole field differ from the usual expressions by suitable phase factors.

Hraskó and Balog (1978) and Frenkel and Hraskó (1977) derive these results by conventional methods. But the introduction of strings and local potentials breaks the symmetry, and the resulting calculations become rather complicated.

The point is that these difficulties are due more to the formalism than to the problem itself, and can be avoided by using suitable tools—differential geometric and fiber bundle techniques. This has been proposed as early as 14 years ago by Souriau and Kostant (Souriau, 1966; Auslander and Kostant, 1967) and became finally accepted by physicists since a series of papers by Wu and Yang (1975, 1976).

We propose here the rederivation of the above results in the framework of symplectic geometry and the geometric quantization theory of Kostant and Souriau (KS theory) (Souriau, 1970; Kostant, 1970; Woodhouse and Simms, 1976). This simplifies a great deal the calculation and explains the

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geometric origin of the results found in Hraskó and Balog (1978) and Frenkel and Hraskó (1977) together with those in Wu and Yang (1976). The basic reference throughout this paper is Souriau's book (1970).

An intuitive interpretation—close to Feynman's path integral ideas—of prequantization was given in Horváthy (1980) (where the relation to the nonintegrable phase factor of Wu and Yang was also discussed). Further details on symmetry transformations and groups will be given elsewhere.

2. THE MONOPOLE'S FIELD CONSTRUCTED BY SYMMETRY ARGUMENTS

The idea followed here—which we have borrowed from Hraskó and Balog (1978)—is very simple: let us send a test particle in the field and try to see what are the limitations on the field imposed by the symmetry properties of the motion of our test particle.

Explicitly, consider a particle moving in $Q=R^3\backslash\{0\}$. In the framework of symplectic mechanics (Souriau, 1970), we describe it by (E,σ), where $E=TQ\times R$ is the evolution space, σ a presymplectic form with $\dim\ker\sigma=1$. The classical motions are the characteristic curves of σ.

According to the principles of mechanics, σ is composed of two parts:

$$\sigma=\sigma_0+e\mathbb{F} \tag{1}$$

$\sigma_0=d\Theta_0=d(m\langle \mathbf{v}, d\mathbf{q}\rangle-m\mathbf{v}^2\,dt)$ describes here a free particle, $\mathbb{F}$ a closed 2-form on space-time, $X=Q\times R$ represents the field, and e is the coupling constant (electric) charge.

A classical symmetry group G (called a dynamical group in this context) is one which acts on E by symplectomorphisms: $g^*\sigma=\sigma$, $g\in G$.

As σ is invariant under time translation and space rotation, these groups become dynamical groups for (E,σ) as soon as $\mathbb{F}$ is invariant under their action on X.

Theorem. The most general form of $\mathbb{F}$ compatible with invariance under time translation and space rotation is

$$\mathbb{F}=\mu\Omega-dU\wedge dt \tag{2}$$

where μ is a real constant, $\Omega_u=\langle u, du\times du\rangle$, the canonical (area) 2-form on $\mathbb{S}_2$, $u\in S_2$, and $U\in C^\infty(\mathbb{R}_+)$.

Physically, $\mu\Omega$ represents the field of a Dirac monopole of strength μ (Souriau, 1970 [2nd edition]; Horváthy, 1980) situated at the origin, and U is a central potential (cf. Hraskó and Balog, 1978).

Proof. As $Q=\mathbb{S}_2\times\mathbb{R}_+$, $\mathbb{F}$—as any 2-form on X—can be written as

$$\mathbb{F}_x=m(x)\cdot\Omega_u+\omega_x\wedge dr+[\varepsilon_x+f(x)\,dr]\wedge dt$$

where $m, f\in C^\infty(X)$, $\omega, \varepsilon\in\Lambda^1(S_2)\otimes C^\infty(\mathbb{R}_+)$.

First, invariance under time translation $t\mapsto t+\tau$ implies that nothing depends on t.

Next, invariance under rotation $(u, r)\mapsto(gu, r)$, $g\in SO(3)$ implies that m, f depend only on r [consequently, $f=-dU/dr$ with $U\in C^\infty(\mathbb{R}_+)$], and ω, ε are rotation invariant.

The condition $d\mathbb{F}=0$ has the consequence

$$\left(\frac{dm}{dr}\Omega-d\omega\right)=0, \qquad d\varepsilon=0$$

Integrating the first equation on the unit sphere around 0, we get $(dm/dr)4\pi=\int_{\mathbb{S}_2}d\omega=0$ by Stoke's theorem. Thus $m=\text{const}=\mu$ and $d\omega=0$. Our theorem follows now from the following lemma.

Lemma. If α is an exact, rotationally invariant 1-form on S_2:

$$\alpha=da, \qquad g^*\alpha=\alpha, \qquad g\in SO(3)$$

then $\alpha=0$.

Proof of the Lemma. Denote by dg the normalized Haar measure on $SO(3)$. Then

$$\alpha=\int_{SO(3)}g^*\alpha\,dg=d\int_{SO(3)}g^*a\,dg=0$$

for $A(u)=\int_{SO(3)}g^*a(u)\,dg$ is constant on S_2. This completes the proof of the lemma.

Indeed, the restriction of ω or ε onto a sphere of radius τ satisfies the conditions of the lemma. ■

Remark. Note that our $\mathbb{F}$ is not necessarily a solution of the classical field equations: we have to impose an additional requirement (corresponding to the second half of the Maxwell equations) of the form (Sternberg, 1978)

$$d(^*\mathbb{F})=J$$

[In our case, this gives $U(r)=k/r$.]

3. CLASSICAL OBSERVABLES AND CONSERVATION LAWS

As a particle moving in the field of a monopole does not have an invariant (under rotations, e.g.) Lagrangian function (Horváthy, 1980) the conventional approach encounters difficulties (cf. Frenkel and Hraskó, 1977). The geometric theory of moments (Souriau, 1970, p. 105), however, works beautifully.

3.1. Angular Momentum. At the classical level, the angular momentum observable is associated with the infinitesimal action of $SO(3)$ on E by the theory of moments.

$so(3)$ is a dynamical group of the system; to a Z from $so(3)$, its Lie algebra, is associated a vector field $\mathbf{Z}_E$. As $so(3)$ is semisimple, we may (by the Killing form) identify both $so(3)$ and its dual to $\mathbb{R}^3$. The moment I is defined by

$$-d(\langle I, Z\rangle)=\mathbf{Z}_E\lrcorner\sigma \tag{3}$$

One checks easily that I exists. The cohomological properties of $SO(3)$ imply (Souriau, 1970; Woodhouse and Simms, 1976) that there is a unique choice for I such that

$$Z\to I_Z:=\langle I, Z\rangle \in C^\infty(E) \tag{4}$$

becomes a Lie algebra isomorphism. [The Lie algebra structure of $C^\infty(E)$ being defined by the Poisson bracket associated with σ.]

If we choose a basis Z_j ($j=1,2,3$) of $so(3)$, the angular momentum observables I_j are defined as $I_j=\langle I, Z_j\rangle$. As a consequence of (4), we have

$$\{I_j, I_k\}=\varepsilon_{jkl}I_l \tag{5}$$

Explicitly, I is computed as

$$I=m\mathbf{q}\times\mathbf{v}-eg\mathbf{u} \tag{6}$$

[$\mathbf{q}=(\mathbf{u}, r)$].

Furthermore, the generalized Noether theorem (Souriau, 1970, p. 107) tells us that I is conserved.

3.2. Energy. Similarly, time translation $t\to t+\tau$ is again, by construction, a dynamical group. The corresponding conserved quantity is

$$E=m\frac{v^2}{2}+U(r) \tag{7}$$

which is, of course, identified with the energy of the system.

4. PREQUANTIZATION

The first step in constructing the quantum system corresponding to (E, σ) is prequantization (Souriau, 1970; Horváthy, 1980; Kostant, 1970; Woodhouse and Simms, 1976). This is possible iff the cohomology class of $\sigma/2\pi$ is integral (we use units where $\hbar=1$). As $e\mathbb{F}$ is essentially the symplectic form of a spinning particle with spin $s=e\mu$ (Souriau, 1970), we get

Proposition. A charged particle moving in the field of a monopole is prequantizable iff

$$n := 2e\mu \in \mathbb{Z} \tag{8}$$

(Dirac's condition). In what follows (8) will always be supposed.

Explicitly, the prequantum manifold $(Y^{(n)}, \omega^{(n)}, \pi^{(n)})$ corresponding to $n\in\mathbb{Z}$—a $U(1)$ principal bundle with connection over E—is constructed as follows:

Let first $2e\mu=1$. Set $\mathbb{S}_3 := \{\zeta\in\mathbb{C}^2 \cdot \bar{\zeta}\zeta=1\}$. (3-sphere of the 4-real-dimensional vector space $\mathbb{C}^2$.) Endow it with the (real) 1-form $\bar{\zeta}(d\zeta/i)$. Define p: $\mathbb{S}_3 \to \mathbb{S}_2$ as

$$[p(\zeta)]^j = \bar{\zeta}\sigma_j\zeta \in \mathbb{R} \qquad (j=1,2,3) \tag{9}$$

with the Pauli matrices σ_j. $U(1)$ acts on $\mathbb{S}_3$: $U(1) \ni z \to \mathbf{z}_1$: $\mathbb{S}_3 \to \mathbb{S}_3 (\zeta \in \mathbb{S}^3) \mathbf{z}_1(\zeta) = z \cdot \zeta$.

Set
$$Y^{(1)} := \{\xi = (u, r, v, t, \zeta) \in TQ \times \mathbb{R} \times \mathbb{S}_3 : p(\zeta) = u\} \tag{10a}$$

$$\pi^{(1)}(\xi) := (u, r, v, t) \in E \tag{10b}$$

$$\omega^{(1)} := \Theta_0 + \bar{\zeta}\,\frac{d\zeta}{i} \tag{10c}$$

$$\mathbf{z}_1(\xi) := (u, r, v, t, \mathbf{z}_1(\zeta)) \tag{10d}$$

Then $(Y^{(1)}, \omega^{(1)}, \pi^{(1)})$ prequantizes (E, σ) for $2e\mu=1$.

For a general $n\in\mathbb{Z}$ use the "méthode de fusion" (Souriau, 1970, p. 326): $\Gamma_n := \{\exp[(2\pi i/n)k]$: $k=0,1,\ldots,|n|\}$ is a discrete subgroup of $U(1)$. Set

$$Y^{(n)} := Y^{(1)}/\Gamma_n \tag{11a}$$

$$\omega^{(n)} := \Theta_0 + n \cdot \bar{\zeta}\,\frac{d\zeta}{i} \tag{11b}$$

$$\pi^{(n)}([\xi]_n) := \pi^{(1)}(\xi) \tag{11c}$$

$$\mathbf{z}_n([\xi]_n) = [\mathbf{z}_1^{1/n}(\xi)]_n \tag{11d}$$

where $[\xi]_n$ is the orbit of $\xi \in Y^{(1)}$ under Γ_n.

Proposition. $(Y^{(n)}, \omega^{(n)}, \pi^{(n)})$ prequantizes (E, σ) for $2e\mu = n \in \mathbb{Z}$.

5. WAVE FUNCTIONS

If we use the "vertical polarization" $q=$const (Souriau, 1970, 2nd edition), the general prescriptions of geometric quantization (Souriau, p. 351), give the following:

Proposition. For $2e\mu = n \in \mathbb{Z}$ a wave function is a complex function

$$\psi^{(n)}: \mathbb{S}_3 \times \mathbb{R}_+ \to \mathbb{C} \tag{12}$$

satisfying

$$\psi^{(n)}(\mathbf{z}_n([\zeta]_n)) = z \cdot \psi^{(n)}([\zeta]_n) \tag{13}$$

Note that $\bar{\psi}^{(n)}\varphi^{(n)}$ is constant [by (13)] on the fibers of the $U(1)$ bundle $(S_3 \times \mathbb{R}_+)/\Gamma_n$; the base space, $S_3 \times \mathbb{R}_+$ admits the rotational-invariant measure $\mathrm{vol}_n \times dr$ (vol_n being the measure belonging to $n\langle u, du \times du\rangle = n\Omega$). Thus

$$\langle \psi^{(n)} | \varphi^{(n)} \rangle := \int_{S_2 \times \mathbb{R}_+} \bar{\psi}^{(n)} \cdot \varphi^{(n)} \mathrm{vol}_n \times dr \tag{14}$$

is a well-defined scalar product on the wave functions.

Alternatively, these wave functions can be viewed as sections of the line bundle associated with our principal $U(1)$ bundle (Kostant, 1970; Woodhouse and Simms, 1976). This is the fact recognized by Wu and Yang (1976).

If we choose a local trivialization $\mathbb{S}_3|_\alpha \sim U_\alpha \times U(1)$ ($U_\alpha \subset \mathbb{S}_2$ being an open set where $\mathbb{F}|_\alpha = F|_{U_\alpha}$ is exact, each wave function will be represented here by an ordinary function $\psi_\alpha^{(n)}: U_\alpha \times \mathbb{R}_+ \to \mathbb{C}$ according to

$$\psi^{(n)}([\zeta]_n)|_\alpha = z_\alpha \cdot \psi_\alpha^{(n)}(u, r) \tag{15}$$

$(u, z_\alpha) \in U_\alpha \times \mathbb{R}_+ \times U(1)$ representing $\zeta \in p^{-1}(U_\alpha \times R_+)$.

If we change our local trivialization to $U_\beta \times \mathbb{R}_+ \times U(1)$, the wave function changes to

$$\psi_\beta^{(n)}(u, r) = C_{\alpha\beta}^{(n)}(u): \psi_\alpha^{(n)}(u, r) \tag{16}$$

$u \in U_\alpha \cap U_\beta$, where $C_{\alpha\beta}^{(n)}: U_\alpha \cap U_\beta \to U(1)$ is the transition function for the bundle $(\mathbb{S}_{3/\Gamma_n}, p, \mathbb{S}_2)$. [This is the gauge transformation in Wu and Yang [1976, p. 367, formulas (8) and (9).]

6. SYMMETRY TRANSFORMATIONS

If we have a classical symmetry transformation a: $E \to E$ which preserves the polarization (a rotation, e.g.) we can express its action on the wave functions as follows:

As E is simply connected, any symplectomorphism a has a prequantum lift, i.e., an A: $Y^{(n)} \to Y^{(n)}$ such that $A^*\omega^{(n)} = \omega^{(n)}$, $\pi^{(n)} \circ A = a \circ \pi^{(n)}$. If the polarization is preserved, A passes to $S_3/\Gamma_n \times \mathbb{R}_+$, and the action on a wave function $\psi^{(n)}$ is simply

$$\hat{a}^{(n)}\psi^{(n)} := \psi^{(n)} \circ A^{-1} \tag{17}$$

If the measure of integration is invariant under a, then (17) defines a unitary operator. Note that A is ambiguous up to a phase factor (Souriau, 1970).

Choosing a local trivialization (17) reads as

$$\left(\hat{a}_\alpha^{(n)}\psi_\alpha^{(n)}\right)(u, r) = {}^{(n)}F_\alpha^a(u) \cdot \psi_\alpha^{(n)}(a^{-1}u, r) \tag{18}$$

where ${}^{(n)}F_\alpha^a(u) : U_\alpha \cap a \circ U_\alpha \to U(1)$ is a unitary factor [cf. Frenkel and Hraskó, 1977 (18)]. (For symplectomorphisms which do not preserve the polarization —time translation, e.g.—the situation is much more complicated.)

If we have now a group of polarization-preserving symplectomorphisms, we can lift all elements individually, but the lifts will not form, in general, a group (Souriau, 1970; Kostant, 1970; Woodhouse and Simms, 1976) (this is just the old problem of unitary versus projective representations). We study here the rotation group $G = SO(3)$.

6.1. Rotations. The action of rotations on the wave function is studied best by working with $SU(2)$, rather than $SO(3)$.

$SU(2)$ is again a dynamical group of our system; its action on (E, σ) is induced by that of $SO(3)$. The same cohomological properties we referred to in Section 3 allow for lifting $SU(2)$ to the prequantum level acting there as a group of quantomorphisms. As the action of $SU(2)$ on E preserves the vertical polarization, we can define its action directly on Q and $\mathbb{S}_3/\Gamma_n \times \mathbb{R}_+$. If we represent a $g \in SU(2)$ by a 2×2 complex matrix, the action of g on S_3/Γ_n is just $[\zeta]_n \mapsto [g\zeta]_n$.

Consequently, the action of $SU(2)$ on the wave function is expressed, according to (17), as

$$\hat{g}\psi^{(n)}([\zeta]_n, r) = \psi^{(n)}\left([g^{-1}\zeta]_n, r\right) \tag{19}$$

As vol_n is a rotationally invariant measure on S_2, we have the following:

Proposition. Equation (19) defines a unitary representation of $SU(2)$.

$SO(3)$ itself will be represented unitarily iff the kernel of the projection $SU(2)\to SO(3)$ goes to identity, which happens iff $n/2=e\mu\in\mathbb{Z}$. In other cases $SO(3)$ will have merely a projective (ray) but not unitary representation (a rotation by 2π, e.g., changes the sign of the wave function).

Expressed in local coordinates we recover the formula (18) in Frenkel and Hraskó (1977):

$$\left(\hat{g}_{\alpha}^{(n)}\psi_{\alpha}^{(n)}\right)(u,r)={}^{(n)}F_{d}^{g}(u)\cdot\psi_{\alpha}^{(n)}(g^{-1}u,r) \tag{20}$$

The phase factor F_d^{α} is found explicitly as follows (cf. Frenkel and Hraskó, 1977): Any $\zeta\in\mathbb{S}_3$ is an image of $\binom{1}{0}\in\mathbb{S}_3$ by a $g_\zeta\in SU(2)$. If g_ζ has Euler angles (φ,θ,ψ) ζ is written as

$$\zeta=\begin{pmatrix}\exp\left[i(\varphi+\psi)/2\right]\cos(\theta/2)\\ i\exp\left[-i(\varphi-\psi)/2\right]\sin(\theta/2)\end{pmatrix}$$

Denote (cf. Frenkel and Hraskó, 1977) $U_d:=\mathbb{S}_2\backslash\{\text{south pole}\}\cdot\cos(\theta/2)\neq 0$ here, and

$$[\zeta]_n|_d\sim\Big(u,\exp\{i[(\varphi+\psi)/2]n\}\Big)\in U_d\times U(1) \tag{21}$$

is a well-defined local trivialization of $\mathbb{S}_3/\Gamma_n$. Thus we get directly [cf. Frenkel and Hraskó, 1977, (18), (19)]

$$ {}^{(n)}F_d^{g}(u)=\exp\left[i(n/2)(\bar{\varphi}+\bar{\gamma}_0-\varphi-\gamma_0)\right] \tag{22}$$

where ζ is any point from $p^{-1}(u)$, having Euler angles $(\varphi,\theta,\gamma_0)$, mapped to $g^{-1}\zeta$ by a $g\in SU(2)$ with Euler angles $(\bar{\varphi},\bar{\theta},\bar{\gamma}_0)$.

6.2. Angular Momentum. The quantum observables called angular momentum are, by definition, the generators of the action of $SU(2)$ on the wave functions. But these generators are, according to (19), deduced from the infinitesimal action of $SU(2)$ on the bundles $\mathbb{S}_3/\Gamma_n\times\mathbb{R}_+$. To a Z from $su(2)$, the Lie algebra of $SU(2)$, is associated a vector field $\mathbf{Z}_n$ on $\mathbb{S}_3/\Gamma_n\times\mathbb{R}_+$. By (19), we have

$$\left(\hat{I}_Z^{(n)}\psi^{(n)}\right)([\zeta]_n,r)=\left(\mathbf{Z}_n\psi^{(n)}\right)([\zeta]_n,r) \tag{23}$$

[The coherence of the notation is a consequence of the cohomological properties of $SU(2)$ allowing for prequantum lifting: the possible ambiguity in choosing a $Z\in su(2)$ to I_Z does not affect the right-hand side of (23).]

As $Z \to \mathbf{Z}_n$ is a Lie algebra isomorphism (the Lie algebra structure on the vector fields on $\mathbb{S}_3/\Gamma_n \times \mathbb{R}_+$ being induced by the Lie bracket), we deduce that

$$I_Z \to \hat{I}_Z^{(n)} \tag{24}$$

is again a Lie algebra isomorphism, for the Lie bracket of $\mathbf{Z}_n$ and $\mathbf{Z}'_n$ is carried by (23) to the commutator of the operators $I_Z, I_{Z'}$.

A local expression for $\hat{I}=(\hat{I}_1, \hat{I}_2, \hat{I}_3)$ is found either by deriving directly (20) or using the equivalence of the principal bundle (Souriau, 1970) and the line bundle (Kostant, 1970; Woodhouse and Sims, 1976) setting; in both cases we get

$$\left(\hat{I}_d^{(n)} = \mathbf{q} \times \left[\frac{1}{i} \frac{\partial}{\partial \mathbf{kju}} - eA_d^{(n)}(\mathbf{q}) \right] - e\mu \cdot \mathbf{u} \right. \tag{25}$$

cf. Frenkel and Hraskó (1977), (23) and Wu and Yang (1976), (12) $[\mathbf{q}=(\mathbf{u}, r)]$.

6.3. Space Reflection. Frenkel and Hraskó (1977) settled the controversy raised by Schwinger (1966) concerning reflection symmetry. As they point out, the correct setting is to study two systems corresponding to motion in the field of monopoles with strengths μ and $-\mu$, respectively. In our terminology we have ${}^{(+)}E = {}^{(-)}E = E = T(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}$ but

$${}^{(+)}\sigma = \sigma_0 + e\mu\Omega \tag{26a}$$

$${}^{(-)}\sigma = \sigma_0 - e\mu\Omega \tag{26b}$$

Space reflection $P(u, r) = (-u, r)$ lifts then to a symplectomorphism P: $(E, {}^{(+)}\sigma_1) \to (E, {}^{(-)}\sigma)$.

The systems $(E, {}^{(+)}\sigma_1)$ and $(E, {}^{(-)}\sigma_2)$ are prequantized to

$${}^{(+)}Y^{(n)} = Y^{(n)} \qquad \text{and} \qquad {}^{(+)}\omega^{(n)} := \omega^{(n)} \tag{27a}$$

$${}^{(-)}Y^{(n)} = Y^{(-n)} = Y^{(n)} = {}^{(+)}Y^{(n)} \qquad \text{and} \qquad {}^{(-)}\omega^{(n)} := \omega^{(-n)} \tag{27b}$$

for $\Gamma_n = \Gamma_{-n}$ [notation (11)].

Consequently, the wave functions of both systems can be identified and have the form

$$\psi^{(n)}: S_3/\Gamma_n \times \mathbb{R}_+ \to \mathbb{C} \tag{28}$$

satisfying the "circulation condition" (13).

In order to study the parity operator on the wave functions, we have to find first a quantomorphism

$$\hat{P}^{(n)}: (Y^{(n)}, \omega^{(n)}) \to (Y^{(n)}, \omega^{(-n)}) \tag{29}$$

projecting to P: $(E, {}^{(+)}\sigma) \to (E, {}^{(-)}\sigma)$. Then, by (17)

$$\hat{P}^{(n)} \psi^{(n)} = \psi^{(n)} \circ \hat{P}^{(-n)} \tag{30}$$

Recall now that $\mathbb{C}^2$ admits a quaternionic structure defined by

$$J\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -\bar{c}_2 \\ \bar{c}_1 \end{pmatrix}, \qquad c_j \in \mathbb{C}, j = 1, 2 \tag{31}$$

One checks at once that J maps $\mathbb{S}_3 \subset \mathbb{C}^2$ to $\mathbb{S}_3 \subset \mathbb{C}^2$, carries $\bar{\zeta}(d\zeta/i)$ to $-\bar{\zeta}(d\zeta/i)$ and its projection to $\mathbb{S}_2$ is just the inversion $u \to -u$. Furthermore, the orbit of a point in $\mathbb{S}_3$ under Γ_n is mapped to the orbit of the corresponding point in $\mathbb{S}_3$ under $\Gamma_{-n} = \Gamma_n$. Thus, up to an arbitrary phase factor $z \in u(1)$,

$$\hat{P}^{(n)}([\zeta]_n, r) = ([J\zeta]_n, r) \tag{32}$$

It is easy to see, that although classically $P^2 = 1$, $[\hat{P}^{(n)}]^2 \neq 1$ in general. Indeed, if $c \in \mathbb{C}$, then $c \circ J = J \circ \bar{c}$, and thus[1]

$$[\hat{P}^{(n)}]^2 \psi^{(n)} = \psi^{(n)} \circ \underline{(-1)}_n^n = (-1)^n \cdot \psi^{(n)} \tag{33}$$

In order to get a local expression, consider the local trivializations (21) corresponding to n; with the choice (32) one gets

$$(\hat{P}_d^{(n)} \psi_d^{(n)})(u, r) = e^{-i\varphi n} \cdot \psi_d^{(n)}(-u, r) \tag{34}$$

cf. Frenkel and Hraskó (1977), equation (27).

7. CONCLUSION

The aim of this paper is to convince physicists that the natural language of studying Dirac's monopole is the geometric one used here. We hope that nobody will, in the future, introduce such complicated and superfluous objects as strings.

[1] $\underline{(-1)}_n^n$ denotes here the action of $(-1)^n \in u(1)$ on $Y^{(n)}$.

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